

Adiabatic response for Lindblad dynamics

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Abstract

We study the adiabatic response of open systems governed by Lindblad evolutions. In such systems, there is an ambiguity in the assignment of observables to fluxes (rates) such as velocities and currents. For the appropriate notion of flux, the formulas for the transport coefficients are simple and explicit and are governed by the parallel transport on the manifold of instantaneous stationary states. Among our results we show that the response coefficients of open systems, whose stationary states are projections, is given by the adiabatic curvature.

1 Introduction

We are interested in extending the theory of adiabatic response of quantum systems undergoing unitary evolution [10, 23] to open (quantum) systems governed by Lindblad evolutions. In particular, we are interested in a geometric interpretation of the response coefficients.

In open systems there is usually some choice in setting the boundary between the system and the bath. Setting the boundary fixes the tensor product structure $\mathcal{H}_s \otimes \mathcal{H}_b$. Choosing a boundary still leaves a residual ambiguity in observables. For example, given a joint Hamiltonian H of the system and the bath, there is no unique way of assigning to H an observable of the form $H_s \otimes \mathbb{1}$ describing the energy of the system alone. (For example, the interaction of an atom with the photonic vacuum has two effects on the

atom: It leads to decay and to “Lamb shift” of the energy levels. One can choose whether to incorporate the Lamb shift in the energy of the atom or in its interaction with the bath.) A further aspect of the ambiguity arises when considering the *flux* (rate) of an observable X . Any assignment $X \mapsto X_s$ of system observables X_s to joint observables X is incompatible with dynamics, if the bath and the system interact. In fact it is generally impossible to satisfy both requirements $X_s \otimes \mathbb{1} \mapsto X_s$ and $\dot{X} = i[H, X] \mapsto \dot{X}_s = i[H_s, X_s]$.

The ambiguity in fluxes is physical and plays a key role in this work. Consider, for example, damped harmonic motion. By Newton, the flux of the momentum is the total force. This force is related to two other forces in this problem:

$$\text{Momentum flux} = \text{Spring force} + \text{Friction force}.$$

The momentum flux can be determined from the trajectory of the particle; The spring force from the force acting on the spring anchor and the friction from the momentum transfer to the bath. All these forces have physical significance and are associated with different measurements. In this work we shall focus on observables that are the analog of the momentum flux.

A time honored strategy to study open systems is to start from a complete Hamiltonian description of the system and the bath [21, 15, 22, 2]. This approach comes at the price of analytical and computational difficulties. Here we choose the complementary “effective” Markovian approach [3, 14] which is often analytically and computationally simpler but is not universally valid. More precisely, we study open systems in the Lindblad framework [14, 13, 17].

The Lindblad operator, denoted \mathcal{L} , is made of a self-adjoint H representing the “energy” of the system and a collection of operators, $\{\Gamma_\alpha\}$, representing the coupling to the bath. The notion of “energy” is, as we have noted, ambiguous and this is manifested in the non-uniqueness of $\{H, \Gamma_\alpha\}$.

We shall call a choice $\{H, \Gamma_\alpha\}$ a *gauge*. Different gauges generate the same dynamics.

We shall consider parametrized Lindbladians, \mathcal{L}_ϕ , where the (classical) parameters $\phi \in \mathcal{M}$ are viewed as controls. This means that $\{H(\phi), \Gamma_\alpha(\phi)\}$ are functions of the controls. \mathcal{M} is the control space, see Fig. 1.

The main focus of this work is the development of an adiabatic¹ response theory for the fluxes; Namely, observables of the form $\dot{X} = \mathcal{L}_\phi^*(X)$. This no-

¹The notion of adiabaticity is contingent on a gap condition, Assumption 2 below.

tion is gauge invariant (independent of the choice $\{H, \Gamma_\alpha\}$). It turns out that the adiabatic response of such observables has several simplifying features.

A different perspective on the choice of observables comes from a gauge invariant formulation of the *principle of virtual work*. For isolated systems the principle of virtual work assigns the observable $\partial_\mu H$ with the variation of the μ -th control ϕ^μ . Since δH is gauge dependent in the Lindblad setting, formulating a gauge invariant notion of the principle of virtual work requires the joint variation $\{\delta H, \delta \Gamma_\alpha\}$.

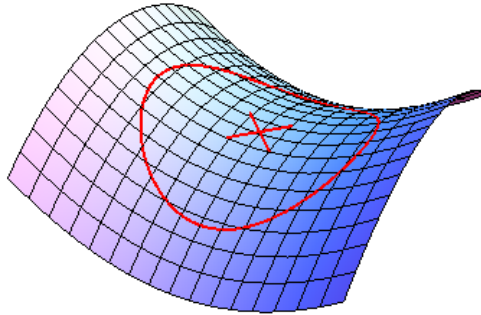


Figure 1: The surface represents the control space \mathcal{M} and the closed curve represents a closed path in the space of controls. The cross denotes a puncture in \mathcal{M} where the adiabatic theory fails and the manifold of stationary states is singular.

Consider a path in control space which is traversed adiabatically, Fig. 1. It is a feature of adiabatic evolutions [4] that stationary states evolve by parallel transport within the manifold of (instantaneous) stationary states. Often this evolution fully determines the response and transport coefficients for fluxes. The transport formulas can be viewed as an analog of Kubo formulas for linear response, emphasizing the geometric flavor.

Although parallel transport captures the geometric aspects of the evolution, as a practical method of calculation of transport coefficients, it suffers from its reliance on solving differential equations. There are, however, two important families of Lindbladians where parallel transport can be determined algebraically, without solving any differential equations. This is the

case for (generic) Lindbladians where the (instantaneous) stationary state is unique. It is also the case for a second family of Lindbladians, namely, dephasing Lindbladians. These share the stationary states with the Hamiltonian evolution [4]. In both cases the transport formulas, the analogs of Kubo formulas, are both geometric and explicit.

In general, the response coefficients of open and closed systems are different. One would like to identify those transport coefficients that are immune to certain mechanism of decoherence and dephasing. For observables of the form $\mathcal{L}^*(X)$ the response coefficients depend on the manifold of stationary states (but not on the underlying dynamics). Immunity then follows whenever the stationary states are unaffected by decoherence and dephasing. This is the case for two physically interesting families of Lindbladians: Dephasing Lindbladians and Lindbladian which allow for decay to the (Hamiltonian) ground state.

2 Lindbladians

The Lindblad (super)² operator [14, 13] is given formally by³

$$\mathcal{L}(\rho) = -i[H, \rho] + \mathcal{D}(\rho), \quad \mathcal{D}(\rho) = \sum_{\alpha} 2\Gamma_{\alpha}\rho\Gamma_{\alpha}^* - \Gamma_{\alpha}^*\Gamma_{\alpha}\rho - \rho\Gamma_{\alpha}^*\Gamma_{\alpha}, \quad (1)$$

where the state ρ is trace class. The Hamiltonian part H is self-adjoint (and local). Γ_{α} are essentially arbitrary. Models describing exchange of energy involve non-self-adjoint Γ_{α} while models of measurement involve Γ_{α} which are spectral projections (non-local in general).

\mathcal{L} is the generator of state and trace preserving contractions. When $\mathcal{D} = 0$ the evolution is unitary. To avoid technical difficulties with unbounded operators we shall assume:

Assumption 1.

- *Weak version: H and Γ_{α} bounded*
- *Strong version: H and Γ_{α} finite dimensional.*

²Super operators will be denoted by script characters.

³The normalization differs by factor 2 from that of [19].

However, we shall also occasionally consider standard physical examples with unbounded operators. We shall study the time evolution of the state ρ :

$$\dot{\rho} = \mathcal{L}(\rho), \quad \left(\dot{\rho} = \frac{d\rho}{dt} \right). \quad (2)$$

2.1 Gauge transformations

\mathcal{L} does not determine $\{H, \Gamma_\alpha\}$. In fact, \mathcal{L} is invariant under the joint variation [13]

$$\delta H = e\mathbb{1} - i \sum_{\alpha} (g_{\alpha}^* \Gamma_{\alpha} - g_{\alpha} \Gamma_{\alpha}^*), \quad \delta \Gamma_{\alpha} = g_{\alpha}, \quad (g_{\alpha} \in \mathbb{C}, e \in \mathbb{R}). \quad (3)$$

Moreover, Γ and $\mathcal{U}\Gamma$ represent the same Lindbladian when \mathcal{U} is unitary in the sense that

$$(\mathcal{U}\Gamma)_{\alpha} = \sum_{\beta} \mathcal{U}_{\alpha\beta} \Gamma_{\beta}, \quad \mathcal{U}^{-1} = \mathcal{U}^*. \quad (4)$$

We shall refer to the freedom in $\{H, \Gamma_{\alpha}\}$ as *gauge freedom*. The observable H , which one would like to interpret as the energy of the system, is therefore ambiguous a priori. However, in the examples we shall consider, one choice is singled out naturally.

2.2 Lindbladians with a unique stationary state

It is convenient to introduce a notation that distinguishes stationary states σ from general states ρ . An important class of Lindbladians are those that have a unique stationary state σ with $\mathcal{L}(\sigma) = 0$ (with $\text{Tr } \sigma = 1$). This is the generic situation in the finite dimensional case. The (super) projections \mathcal{P} on the stationary state is given by

$$\mathcal{P}(\rho) = \sigma \text{Tr } \rho, \quad (\text{Tr } \sigma = 1). \quad (5)$$

Evidently \mathcal{P} is trace preserving and $\mathcal{P}^2 = \mathcal{P}$ (since $\text{Tr } \sigma = 1$). It is *not orthogonal*, not even formally. In fact, the dual projection \mathcal{P}^* , which acts naturally on observables, is given by a different expression,

$$\mathcal{P}^*(X) = \mathbb{1} \cdot \text{Tr}(X\sigma). \quad (6)$$

A basic identity we shall need is

$$\mathcal{L}\mathcal{P} = 0 = \mathcal{P}\mathcal{L}. \quad (7)$$

The first equality is evident. The second follows from

$$\mathcal{P}\mathcal{L}\rho = \sigma\text{Tr}(\mathcal{L}(\rho)) = 0.$$

We shall denote by \mathcal{Q} the complementary projection

$$\mathcal{P} + \mathcal{Q} = \mathbb{1}. \quad (8)$$

Evidently,

$$\mathcal{L}\mathcal{Q} = \mathcal{L} = \mathcal{Q}\mathcal{L}. \quad (9)$$

This will play a role in the sequel.

2.3 Dephasing Lindbladians

Dephasing Lindbladians are intermediate between the unitary family and Lindbladians with a unique stationary state. They are characterized by $\Gamma_\alpha = \Gamma_\alpha(H)$ for some functions Γ_α . In particular,

$$\mathcal{L}(P) = 0 = i[H, P], \quad (10)$$

where P is a spectral projection for H .

In the finite dimensional case all spectral projections P_j are finite dimensional and dephasing Lindbladians share the stationary states with the Hamiltonian. The manifold of stationary states is then the span of the P_j . The (super) projections \mathcal{P} on this manifold and its complement \mathcal{Q} , are given by

$$\mathcal{P}(\rho) = \sum_j P_j \rho P_j, \quad \mathcal{Q}(\rho) = \sum_{j \neq k} P_j \rho P_k. \quad (11)$$

\mathcal{P} and \mathcal{Q} are orthogonal projections. \mathcal{P} satisfies Eq. (7) and \mathcal{Q} satisfies Eq. (9).

3 Stationary states

Let us consider the (super) projections \mathcal{P} on the manifold of stationary states from a perspective that puts the special classes treated above in a uniform context.

It is a basic property of Lindblad operators that [4]

$$\text{Ker } \mathcal{L} \cap \text{Range } \mathcal{L} = \{0\}. \quad (12)$$

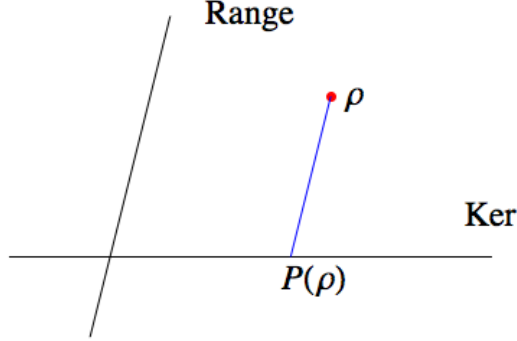


Figure 2: The (super) projection \mathcal{P} projects on $\text{Ker } \mathcal{L}$ along $\text{Range } \mathcal{L}$. The two spaces are transversal in the Lindblad setting.

This follows from $\exp(t\mathcal{L})$ being a contraction: $\mathcal{L}^2\rho = 0$ implies $\exp(t\mathcal{L})\rho = \rho + t\mathcal{L}\rho$ and we conclude $\mathcal{L}\rho = 0$.

This allows to define a projection on $\text{Ker}(\mathcal{L}) \oplus \text{Range}(\mathcal{L})$ by

$$\mathcal{P}\rho = \begin{cases} \rho & \text{when } \rho \in \text{Ker}(\mathcal{L}) \\ 0 & \text{when } \rho \in \text{Range}(\mathcal{L}). \end{cases} \quad (13)$$

Assumption 2 (Gap condition). *0 is an isolated point in the spectrum of \mathcal{L} and \mathcal{P} is given by the Riesz projection*

$$\mathcal{P} = \frac{1}{2\pi i} \oint \frac{dz}{z - \mathcal{L}}, \quad (14)$$

where the contour encircles 0 but no further points of the spectrum (see Fig. 3).

Remark 1. *If 0 is an eigenvalue of finite algebraic multiplicity, then the Riesz projection part is for free. The assumption is implied as a whole by the strong version of Assumption 1. The assumption guarantees that $\text{Range } \mathcal{L}$ is a closed subspace.*

The consistency of Eqs. (13) and (14) deserves a discussion. In fact, the Riesz projection \mathcal{P} always satisfies the first line of Eq. (13) and the validity of the second one is the core of the assumption. To see this consider, besides of \mathcal{P} given by Eq. (14), also \mathcal{Q} similarly given in terms of a contour encircling the complementary part of the spectrum. Then

$$\mathcal{P} + \mathcal{Q} = \mathbb{1}, \quad [\mathcal{P}, \mathcal{L}] = [\mathcal{Q}, \mathcal{L}] = 0, \quad (15)$$

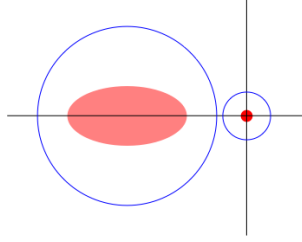


Figure 3: Since \mathcal{L} is a contraction, its spectrum away from the origin is contained in the ellipsoid blob in the left half-plane. The origin is assumed to be an isolated point in the spectrum. The circles are the integration contour for the Riesz projection \mathcal{P} and \mathcal{Q} .

proving the first line. Now assuming the second line of Eq. (13), the eigenspace associated to 0 has a trivial Jordan block. This means that the Laurent expansion of the resolvent does not have a z^{-2} term and hence

$$\mathcal{P}\mathcal{L} = \frac{1}{2\pi i} \oint dz \frac{z}{z - \mathcal{L}} = 0 \quad (16)$$

and

$$\mathcal{Q}\mathcal{L} = \mathcal{L}, \quad \mathcal{L}^* = \mathcal{L}^* \mathcal{Q}^*. \quad (17)$$

This places Eqs. (7, 9) into their general natural context.

Example 1 (Gapless Lindbladians). *It may happen that H is gapped and \mathcal{L} is gapless. For example, let H be a Hamiltonian with a ground state separated by a gap from a continuous spectrum. Then the associated Lindbladian $\mathcal{L}\rho = -i[H, \rho]$ has the eigenvalue 0 embedded in the continuous spectrum.*

4 Controlled Lindbladians

The geometric aspects emerge when one turns one's attention to a parametrized family of Lindbladians \mathcal{L}_ϕ . We shall call the parameters $\phi \in \mathcal{M}$ *control* and \mathcal{M} the control space. This makes the Hamiltonian $H(\phi)$ and the coupling to the bath $\Gamma_\alpha(\phi)$ functions of the controls. The explicit form of these functions is, of course, model specific.

Assumption 3 (Controlled Lindbladians).

- (A) The Lindbladian \mathcal{L}_ϕ is a bounded (super) operator which is a smooth function of the controls ϕ .
- (B) The gap condition, Assumption 2, holds for all \mathcal{L}_ϕ .

4.1 Iso-spectral Lindbladians

A distinguished family of controlled Lindbladians is the family of iso-spectral Lindbladians given by the action of unitaries on H and Γ :

$$H(\phi) = U(\phi) H U^*(\phi), \quad \Gamma_\alpha(\phi) = U(\phi) \Gamma_\alpha U^*(\phi). \quad (18)$$

The Lindbladian describing a harmonic oscillator coupled to a thermal bath, whose anchoring point is controlled, is an example:

Example 2 (Controlled oscillator in thermal contact). *A Harmonic oscillator, anchored at the origin and coupled to a heat bath, is described by the Lindbladian*

$$H = a^* a, \quad \Gamma_- = \sqrt{\gamma_-} a, \quad \Gamma_+ = \sqrt{\gamma_+} a^*, \quad (\gamma_- > \gamma_+ > 0) \quad (19)$$

and where $\sqrt{2}a = x + ip$. The stationary state of the oscillator is a thermal state with $\beta = \log(\gamma_-/\gamma_+)$ [13]. The Harmonic oscillator with controlled anchoring point is described by the iso-spectral family with $U(\phi) = e^{-ip\phi}$. Explicitly

$$\sqrt{2} a(\phi) = \sqrt{2} U(\phi) a U^*(\phi) = (x - \phi) + ip. \quad (20)$$

Since the Γ 's adjust to H the oscillator wants to relax to the thermal state of the instantaneous Hamiltonian.

4.2 Parallel transport

We shall denote instantaneous stationary states by σ . By definition $\mathcal{P}(\sigma) = \sigma$ (we allow $\dim \mathcal{P} \geq 1$). By Assumption 3 the projection on the stationary states, \mathcal{P}_ϕ , is a smooth projection on control space and $d\mathcal{P}$ is a bounded operator valued form.

The differential of $\sigma = \mathcal{P}\sigma$ gives the identity $d\sigma = (d\mathcal{P})\sigma + \mathcal{P}d\sigma$. Since \mathcal{P} is a projection $\mathcal{P}(d\mathcal{P})\mathcal{P} = 0$ and consequently $\mathcal{Q}(d\sigma) = (d\mathcal{P})\sigma$, while $\mathcal{P}(d\sigma)$ remains undetermined. Parallel transport is the requirement, given in two equivalent forms,

$$\mathcal{P}(d\sigma) = 0, \quad d\sigma = (d\mathcal{P})\sigma. \quad (21)$$

This evolution of σ is indeed interpreted geometrically as parallel transport: There is no motion in \mathcal{P} . The case $\dim \mathcal{P} = 1$ is a special simple case, in that there is a unique state $\sigma = \sigma(\phi)$ in the range of $\mathcal{P}(\phi)$. It solves Eq. (21) without further ado.

Proposition 1. *Under Assumption 3, the form $d\sigma$ is trace class.*

Proof. Follows from Eq. (21), the fact that σ is trace class and the boundedness of $d\mathcal{P}$. \square

4.3 Holonomy of parallel transport

In general, parallel transport, Eq. (21), does *not* integrate to a function on control space \mathcal{M} unless the curvature vanishes: $\mathcal{P}d\mathcal{P} \wedge d\mathcal{P} = 0$ (see Appendix A). If such a function $\sigma = \sigma(\phi)$ exists, it will be called an *integral of parallel transport*. This is, of course, automatic if either $\mathcal{M} = \mathbb{R}$, or $\dim \mathcal{P} = 1$, (see Eq. (5)).

Parallel transport is consistent with the convex structure of stationary states [4]. As a consequence it preserves extremal stationary states. Recall that a (stationary) state is called *extremal* if it can not be written as a convex combination of two other (stationary) states. For such extremal states we have:

Proposition 2 (Parallel transport of extremal states). *The parallel transport equation takes extremal stationary state to an extremal stationary state. If, moreover the manifold of (instantaneous) stationary states is a simplex, spanned by a finite number of isolated extremal states $\sigma_j(\phi)$, then the function on \mathcal{M}*

$$\sum p_j \sigma_j(\phi) \tag{22}$$

with p_j independent of ϕ , is an integral of parallel transport.

Proof. A more general statement has been proved in [4, Proposition 3]. The intuition is that states move inside $\text{Ker } \mathcal{L}$ as little as possible. In particular the boundary should be mapped by parallel transport to the boundary and extremal points to extremal points. \square

Parallel transport is path independent for two important families of Lindbladians:

- Lindbladians with a unique stationary state, (Section 2.2).

- Dephasing Lindbladians where the isolated extremal states are the one dimensional spectral projections $\sigma_j(\phi) = P_j(\phi)$, (Section 2.3).

Finally we discuss the parallel transport for iso-spectral families.

Proposition 3. *The unitary family $\sigma(\phi) = U(\phi)\sigma U^*(\phi)$, where $Gd\phi = iU^*dU$, is an integral of parallel transport, Eq. (21), if and only if*

$$\mathcal{P}([G, \sigma]) = 0. \quad (23)$$

In particular this is the case when σ is an isolated extremal point.

Proof. Condition (23) follows by inserting

$$d\sigma = -i[G, \sigma]d\phi \quad (24)$$

into Eq. (21). The last claim is a consequence of Prop. 2. \square

Example 3. *Condition (23) holds for any iso-spectral family with a unique stationary state (and G bounded). In this case \mathcal{P} is given by Eq. (5) and*

$$\mathcal{P}([G, \sigma]) = \sigma \text{Tr}([G, \sigma]) = 0$$

by the cyclicity of the trace.

5 Observables and fluxes

We denote observables by X . The evolution of observables (in the Heisenberg representation) is generated by \mathcal{L}^* :

$$\dot{X} = \partial_t X + \mathcal{L}^*(X), \quad \left(\dot{X} = \frac{dX}{dt} \right) \quad (25)$$

where

$$\mathcal{L}^*(X) = i[H, X] + \mathcal{D}^*(X), \quad \mathcal{D}^*(X) = \sum_{\alpha} \Gamma_{\alpha}^*[X, \Gamma_{\alpha}] + [\Gamma_{\alpha}^*, X]\Gamma_{\alpha}. \quad (26)$$

\dot{X} is itself an observable: We refer to \dot{X} either as the *flux* (or rate) of X or simply as the flux \dot{X} . For example, the velocity is the flux of the position and the force is the flux of the momentum.

Assumption 4. X is not explicitly time dependent, i.e. $\partial_t X = 0$, and hence $\dot{X} = \mathcal{L}^*(X)$.

Fluxes lie in $\text{Range } \mathcal{L}^*$. They have the special feature of vanishing expectation in stationary states. In fact:

Proposition 4 (No currents). *Let σ be a trace class stationary state and \dot{X} the flux of the bounded observable X with H and Γ_α bounded. Then, $\text{Tr}(\dot{X}\sigma) = 0$. Conversely, if $\text{Tr}(A\sigma) = 0$ for any such stationary state, then $A = \mathcal{L}^*(X) = \dot{X}$ for some bounded observable X .*

Proof. The (super) operator \mathcal{L}^* acts on the space of bounded operators, this being the dual of the space of trace class operators. By Eq. (25)

$$\text{Tr}(\dot{X}\sigma) = \text{Tr}(\mathcal{L}^*(X)\sigma) = \text{Tr}(X\mathcal{L}(\sigma)) = 0. \quad (27)$$

Conversely when (a bounded) A has vanishing expectation in stationary states, then

$$0 = \text{Tr}(A\mathcal{P}(\rho)) = \text{Tr}(\mathcal{P}^*(A)\rho) \quad (28)$$

holds for any ρ . Hence $\mathcal{P}^*(A) = 0$ and A lies in the range of \mathcal{L}^* . \square

An example of an observable which is not a flux is:

Example 4 (Hopping on a clock). *Consider a quantum particle on a ring with p sites, \mathbb{Z}_p , evolving by the (bounded) Hamiltonian*

$$H(\phi) = e^{i\phi}T + e^{-i\phi}T^*, \quad (T\psi)(n) = \psi(n-1).$$

$p\phi$ may be interpreted as the magnetic flux threading the ring. The stationary states are $\langle n | \sigma_j | m \rangle = p^{-1} e^{i(n-m)2\pi j/p}$, ($j = 0, \dots, p-1$). The angular velocity is the (bounded) operator

$$-\partial_\phi H = -i(e^{i\phi}T - e^{-i\phi}T^*),$$

Since $-\text{Tr}(\partial_\phi H \sigma_j) = 2 \sin(2\pi j/p - \phi)$ does not vanish in stationary states, $\partial_\phi H$ is not a flux: The angle is not an observable.

The velocity is the flux of the position operator, which is usually unbounded. It is therefore interesting to examine conditions that would allow extending Prop. 4 to unbounded operators. Indeed, the gap condition implies that the expectation values of fluxes vanish in stationary states even for

unbounded X provided $Q^*(X)$ is bounded. Indeed, the gap condition allows us to use Eq. (17) and replace Eq. (27) by

$$\mathrm{Tr}(\mathcal{L}^*(X)\sigma) = \mathrm{Tr}(\mathcal{L}^*(Q^*(X))\sigma) = \mathrm{Tr}(\mathcal{Q}^*(X)\mathcal{L}(\sigma)) = 0. \quad (29)$$

A more careful discussion of this point is given in Prop. 16 of Appendix B. Historically, equilibrium currents in superconductors and mesoscopic systems were a matter of debate, see e.g. [11, 12].

An example where X is unbounded but $Q^*(X)$ is bounded is:

Example 5 (Taming X). *Consider a quantum particle with spin hopping on the integer lattice. The Hilbert space is $\ell(\mathbb{Z}) \otimes \mathbb{C}^2$ and let the Hamiltonian be*

$$H = T \otimes a + T^* \otimes a^*,$$

where T is the unit left shift and a and a^* are the spin lowering and raising operators $a^2 = (a^*)^2 = 0$, $\{a, a^*\} = 1$. Since $H^2 = \mathbb{1}$ we can write H as a difference of two (infinite dimensional) projections:

$$H = P_+ - P_-, \quad 2P_{\pm} = \mathbb{1} \pm H.$$

The position operator is $X \otimes \mathbb{1}$. It is clearly unbounded. Since $[T, X] = T$, the velocity is the bounded operator

$$\dot{X} = i[H, X \otimes \mathbb{1}] = i(T \otimes a - T^* \otimes a^*).$$

In fact, $\dot{X}^2 = \mathbb{1}$. The appropriate version of Eq. (11) says that Q^* is given by

$$\begin{aligned} \mathcal{Q}^*(X) &= P_+ X P_- + P_- X P_+ \\ &= P_+[X, P_-] + P_-[X, P_+] \\ &= \frac{1}{2} (-P_+[X, H] + P_-[X, H]) \\ &= -\frac{i}{2} (P_+ - P_-) \dot{X}, \end{aligned}$$

being a product of bounded operators, it is bounded, even though X is not.

5.1 Virtual work for Lindbladians

The principle of virtual work associates observables with variations of a controlled Hamiltonian $H(\phi)$ (see Section 4). Our aim here is to formulate a corresponding principle for Lindbladians.

Observe first that \mathcal{L} is a (super) operator, so its variation *does not* define an observable and moreover, the notion of “energy” is ambiguous in Lindblad evolutions. The principle of virtual work we formulate is gauge invariant in the sense of Section 2.1.

Theorem 5. *The observables X_μ given by the joint variation of H and Γ*

$$X_\mu \delta \phi^\mu = \delta H + i \sum_\alpha (\Gamma_\alpha^* \delta \Gamma_\alpha - \delta \Gamma_\alpha^* \Gamma_\alpha) \quad (30)$$

are (formally) self-adjoint and free from the ambiguity in H and Γ , under ϕ independent gauge transformations.

Proof. For g_α and e independent of ϕ , the gauge transformation, Eq. (3), affects the variation by

$$(\delta H, \delta \Gamma_\alpha) \rightarrow \left(\delta H - i \sum_\alpha (g_\alpha^* \delta \Gamma_\alpha - g_\alpha \delta \Gamma_\alpha^*), \delta \Gamma_\alpha \right).$$

This leaves X_μ invariant. The same applies to the transformation (4). \square

Remark 2 (A second family). *A second family of observables that are gauge invariant is $\delta(\sum_\alpha [\Gamma_\alpha, \Gamma_\alpha^*])$.*

The observables X_μ extend the notion of the principle of virtual work to the Lindbladian setting. The physical interpretation of X_μ is often suggested by dimensional analysis. However, their interpretation depends on the precise choice of controls and is model dependent.

Example 6 (Controlled oscillator in thermal contact: Example 2 continued). *Shifting the anchoring point of the oscillator gives*

$$\delta H = -\frac{a + a^*}{\sqrt{2}} \delta \phi = -x \delta \phi, \quad i(\Gamma_\pm^* \delta \Gamma_\pm - \delta \Gamma_\pm^* \Gamma_\pm) = \mp \gamma_\pm \frac{a^* - a}{i\sqrt{2}} \delta \phi = \pm \gamma_\pm p \delta \phi.$$

$-x$ is the spring force, while $-\gamma_- p$ gives the friction force due to the cold contact and $\gamma_+ p$ the gain from the hot contact. The total force is the momentum flux

$$\dot{p} = -x - (\gamma_- - \gamma_+) p, \quad (\gamma_- > \gamma_+). \quad (31)$$

In the example, the principle of virtual work gives a flux. This is not a coincidence. For iso-spectral Lindbladians, Eq. (18), virtual work is a flux. More precisely, let G_μ denote the (local) infinitesimal generators

$$G\delta\phi = G_\mu\delta\phi^\mu = iU^*\delta U \quad (32)$$

(summation implied). The variations are:

$$\delta H = i[H, G_\mu]\delta\phi^\mu, \quad \delta\Gamma_\alpha = i[\Gamma_\alpha, G_\mu]\delta\phi^\mu. \quad (33)$$

Theorem 6 (Virtual work and fluxes). *For iso-spectral families of Lindbladians generated by G_μ , the observables associated with the principle of virtual work, Eq. (30), are the fluxes of the generators G_μ :*

$$\mathcal{L}^*(G_\mu)\delta\phi^\mu = \delta H + i \sum_\alpha (\Gamma_\alpha^* \delta\Gamma_\alpha - \delta\Gamma_\alpha^* \Gamma_\alpha). \quad (34)$$

In particular we have Noether's theorem in the form: If δU is a symmetry, in the sense that the r.h.s. vanishes, then its generator is a conserved quantity.

Proof. Transparent. □

5.2 Currents

Just as there are several notions of force in a damped oscillator, there are several notions of currents in an open system. In an open system charge need not be conserved as charge may migrate between the system and the bath. One can distinguish three notions of currents, related by one relation: The rate of charge in a subsystem; the rate of charge transfer between a subsystems and its complement, and the rate of charge transferred from the subsystem to the bath.

More precisely, consider a subsystem associated with a domain Ω . Let Q_Ω denote the observable associated with the charge in Ω . The first notion of a current is the flux of Q_Ω , namely,

$$\dot{Q}_\Omega = \mathcal{L}^*(Q_\Omega) = i[H, Q_\Omega] + \mathcal{D}^*(Q_\Omega) \quad (35)$$

\dot{Q}_Ω , being a flux, is gauge invariant in the sense that it depends only on \mathcal{L} and not on its partition into H and Γ . It is, however, *non-local* in general since Γ_α , in contrast with H , need not be local.

We assume that H is charge conserving, i.e. $[H, Q_{\Omega \cup \Omega^c}] = 0$. This allows to define the current from the subsystem Ω to its complement by

$$I_{\partial\Omega} = i[H, Q_{\Omega}]. \quad (36)$$

Since H is local, I_{Ω} is localized near the boundary, $\partial\Omega$, between the subsystem and its complement. However, it depends on the partitioning of \mathcal{L} into H and Γ_{α} .

The remaining term gives the rate of dissipated charge, S_{Ω} . By definition, the three currents are related by

$$\dot{Q}_{\Omega} = I_{\partial\Omega} + S_{\Omega}. \quad (37)$$

The three currents have different characters and are measured by different instruments. \dot{Q}_{Ω} is measured by an electrometer while $I_{\partial\Omega}$ can be measured by an ammeter that monitors the flow at the boundary between the subsystems. The last term has been called dissipative current in [8, 16].

Example 7. *Consider Fermions hopping on a one dimensional lattice which can also tunnel in and out of a bath. The Lindbladian has*

$$H = \sum_j (a_{j+1}^* a_j + a_j^* a_{j+1} + \mu a_j^* a_j), \quad \Gamma_j^- = \sqrt{\gamma_-} a_j, \quad \Gamma_j^+ = \sqrt{\gamma_+} a_j^*$$

with a_j the usual Fermion annihilation operators for site j . The charge in the left semi-infinite box is

$$Q_L = \sum_{j \leq 0} a_j^* a_j$$

and the currents in Eq. (37) are

$$I_{\partial L} = i(a_1^* a_0 - a_0^* a_1), \quad S_L = 2 \sum_{j \leq 0} (\gamma_+ a_j a_j^* - \gamma_- a_j^* a_j) :$$

$I_{\partial L}$ is localized at the boundary of the box, whereas S_L is not.

Dissipating currents can arise also when the Lindbladian is charge conserving as we discuss below.

5.3 Currents in a magnetic field

The action of a magnetic field on charged particles endows the dynamics with chirality. This has interesting consequences for currents. Consider the Lindbladian describing a charged particle in the plane under the influence of a constant magnetic field coupled to a heat bath. The Hamiltonian is the Landau Hamiltonian⁴:

$$H = D^*D, \quad D = -i\partial_1 + \partial_2 + Bx_2 \equiv v_1 + iv_2, \quad (v_\mu = v_\mu^*), \quad (38)$$

and the (Markovian) thermal bath, is described by (cf. Example 2))

$$\Gamma_- = \sqrt{\gamma_-}D, \quad \Gamma_+ = \sqrt{\gamma_+}D^*, \quad (\gamma_- > \gamma_+ \geq 0). \quad (39)$$

We shall call the generator of the corresponding evolution a thermal Landau Lindbladian.

Proposition 7. *The (total) current density of the Landau Lindbladian of Eqs. (38, 39) is*

$$j_\mu(x_0) = \{\rho(x_0), v_\mu\} - (\gamma_+ + \gamma_-)\partial_\mu\rho(x_0) + 2(\gamma_- - \gamma_+)\rho(x_0)\varepsilon_{\mu\nu}v_\nu. \quad (40)$$

The charge density is $\rho(x_0) = \delta(\cdot - x_0)$ and $\varepsilon_{\mu\nu}$ is the completely anti-symmetric (Levi-Civita) tensor. The (total) current satisfies charge conservation:

$$\partial_t\rho = -\partial_\mu j_\mu. \quad (41)$$

Before proving the statement, let us comment about its content. The Hamiltonian current is proportional and parallel to the velocity $2v_\mu$. The dissipative current has a (non-chiral) diffusive term proportional to the gradient of the density and a further chiral term. The dissipative currents can be interpreted in terms of Brownian motion (see below).

Proof. The dissipative terms of the Lindbladian are

$$\mathcal{D}_\pm^*(X) = \Gamma_\pm^*[X, \Gamma_\pm] + [\Gamma_\pm^*, X]\Gamma_\pm.$$

For a function $X = f(x_1, x_2)$ of position we have

$$i[H, f] = \{v_\mu, \partial_\mu f\}, \quad \mathcal{D}_\pm^*(f) = \gamma_\pm \Delta f \mp 2\gamma_\pm (\partial_\mu f)\varepsilon_{\mu\nu}v_\nu. \quad (42)$$

⁴Strictly speaking, this example lies outside the scope of our framework since it involves unbounded operators. However, it is simple enough that one can check that formal manipulations are indeed justified.

Eq. (35) then gives the dual form of the statements of the proposition, namely,

$$\partial_t f = \int j_\mu(x_0) \partial_\mu f(x_0) d^2 x_0, \quad f = \int \rho(x_0) f(x_0) d^2 x_0.$$

□

Stochastic interpretation

The dissipative currents admit an interpretation in terms of a (classical) stochastic process. To see this note first that for functions $X = f(v_\mu)$ of either velocity ($\mu = 1, 2$)

$$\mathcal{D}_\pm^*(f) = \gamma_\pm B^2 f'' \pm 2\gamma_\pm B f' v_\mu \quad (43)$$

which can be read as if originating from a (exciting or damping) Langevin equation

$$dv_\mu = \pm 2\gamma_\pm B v_\mu dt + B db_{\mu,t},$$

where b_μ is a Brownian motion with zero drift and variance

$$\mathbb{E}(db_{\mu,t} db_{\nu,t}) = 2\gamma_\pm \delta_{\mu\nu} dt.$$

In fact, expanding $\mathbb{E}(f(v_\mu + dv_\mu))$ to first order in dt and to second order in db_t yields that expression.

To derive the Langevin equation for dx we first note that the guiding center (r_1, r_2) ,

$$r_\mu = x_\mu + \frac{\varepsilon_{\mu\nu}}{B} v_\nu \quad (44)$$

satisfies $[r_\mu, v_\nu] = 0$ and thus is a constant of motion for the Lindbladian, $\mathcal{L}^*(r_\mu) = 0$. Insisting on r_μ being a constant of motion, we have

$$dx_\mu = -\frac{\varepsilon_{\mu\nu}}{B} dv_\nu = \varepsilon_{\mu\nu} (\mp 2\gamma_\pm v_\nu dt - db_{\nu,t}). \quad (45)$$

In view of $\mathbb{E}(\varepsilon_{\mu\nu} db_{\nu,t} \varepsilon_{\mu'\nu'} db_{\nu',t}) = 2\gamma_\pm \delta_{\mu\mu'} dt$ this is the Langevin equation corresponding to Eq. (42). (Beware: $dx_\mu \neq v_\mu dt$.)

We can now combine $\mathcal{L}^*(r_\mu) = 0$ with Theorem 6 to conclude

Proposition 8 (Iso-spectral Landau Lindbladians). *The velocity, \dot{x} , is the (negative) virtual work associated with the iso-spectral family of Landau Lindbladians $\{H(\phi), \Gamma_\pm(\phi)\}$ generated by*

$$G_\mu = B^{-1} \varepsilon_{\mu\nu} v_\nu. \quad (46)$$

The unitary acts on wave functions by

$$(U_\phi\psi)(x_1, x_2) = \psi(x_1 + \phi_2/B, x_2 - \phi_1/B)e^{i\phi_2x_2}. \quad (47)$$

The physical interpretation emerges by noting that

$$U_\phi v_\mu U_\phi^* = v_\mu - \phi_\mu. \quad (48)$$

Since ϕ_μ appears in H like a pure gauge field, its variation in time, $-\dot{\phi}$ is a constant electric field that drives the system. Alternatively, the proposition may be viewed as a manifestation of gauge and translation covariance, in the sense that $-i\partial_\mu$ and x_μ appear in the Lindbladian only through the minimal coupling expression v_μ . The variations in Eq. (34) generated by $-G_\mu$ and x_μ are then the same by

$$-\frac{\partial}{\partial\phi_\mu}U_\phi v_\mu U_\phi^* = i[v_\nu, x_\mu];$$

in fact both sides equal $\delta_{\mu\nu}$.

6 Adiabatic Response

We are interested in adiabatically changing controls; $\phi = \phi(s)$ where $s = \varepsilon t$ is the slow time. The evolution equation for the state ρ is

$$\varepsilon \frac{d\rho}{ds} = \mathcal{L}_\phi \rho. \quad (49)$$

with initial state that is an instantaneous equilibrium state. A key feature of adiabatic theory is that the evolution of ρ is slaved to the evolution of σ . We borrow from [4]:

Proposition 9 (Adiabatic evolution). *Under Assumption 3 the solution of Eq. (49) with initial condition the stationary state $\sigma(0)$ is*

$$(\mathcal{P}\rho)(s) = \sigma(s) + \begin{cases} O(\varepsilon^\infty) & \text{if } \dim \mathcal{P} = 1 \\ O(\varepsilon) & \text{if } \dim \mathcal{P} \geq 2; \end{cases} \quad (50)$$

and

$$(\mathcal{Q}\rho)(s) = \varepsilon \mathcal{L}^{-1} \dot{\sigma}(s) + O(\varepsilon^2), \quad (51)$$

where $\sigma(s)$ is the corresponding integral of parallel transport.

$\mathcal{L}^{-1}(\dot{\sigma})$ is well defined and bounded since $\dot{\sigma} \in \text{Range } \mathcal{L}$. This follows from parallel transport $\dot{\sigma} = \mathcal{Q}\dot{\sigma}$ and the definition of \mathcal{Q} as the projection on $\text{Range } \mathcal{L}$.

6.1 The response of unique stationary states

We are interested in the response of the observable X of an adiabatically driven system. The case of a unique stationary state is simpler than the general case and we treat it first.

Proposition 10 (Response coefficients). *Suppose Assumption 3 holds and the stationary state is unique. Let X be a bounded observable and ρ a solution of the adiabatic Lindblad evolution, Eq. (49), with initial state a normalized stationary state $\sigma(0)$. Then, the response at slow time s is memory-less and is given by*

$$\mathrm{Tr}(X\rho(s)) = \mathrm{Tr}(X\sigma(\phi)) + \varepsilon F_\nu(\phi) \dot{\phi}^\nu + O(\varepsilon^2),$$

(summation implied) with $\sigma(\phi)$ the instantaneous stationary state and $\phi = \phi(s) \in \mathcal{M}$. The response coefficients

$$F_\nu(\phi) = \mathrm{Tr}(X\mathcal{L}_\phi^{-1}(\partial_\nu\sigma(\phi))) \quad (52)$$

are functions on control space \mathcal{M} .

Proof. This is a direct consequence of the adiabatic expansion, Eq. (50). $\partial_\nu\sigma$ is trace class by Prop. 1. \square

The first term $\mathrm{Tr}(X\sigma)$ is of $O(1)$, and describes the *persistent response*, a property of the stationary state. The second term is the *driven response* which is proportional to the driving $\varepsilon\dot{\phi}$, the (unscaled) velocity of the controls.

One is often interested in situations where $F(\phi)$ is constant on \mathcal{M} . This feature depends on additional structure (e.g. thermodynamic limit, disorder [9, 6, 1]). Observe that the expression for $F(\phi)$ involves inverting \mathcal{L} , an operator with a non-trivial kernel, and so is not completely elementary.

Remark 3. *The formula for the response coefficient F_ν can be cast in a way that is formally reminiscent of Kubo's formula:*

$$F_\nu(\phi) = -\lim_{\varepsilon \downarrow 0} \int_0^\infty dt \mathrm{Tr}((e^{(\mathcal{L}_\phi^* - \varepsilon)t} X) (\partial_\nu\sigma(\phi))).$$

7 Response of fluxes

When the manifold of stationary states is multidimensional, the persistent response has memory and F can not be viewed anymore as functions on control space \mathcal{M} (see Section 4.3). However, in the case of observable which are fluxes several simplifications occur: There is no persistent response and the formula for F_ν simplifies and becomes elementary. If, in addition, the extremal stationary states are isolated (Section 4.3) then, in addition, F defines a function on \mathcal{M} .

By definition, a flux (which is not explicitly time dependent, Assumption 4) can be written as

$$\varepsilon \dot{X} = \mathcal{L}^*(X) = \mathcal{L}^*(\mathcal{Q}^*X), \quad \left(\dot{X} = \frac{dX}{ds} \right), \quad (53)$$

where the replacement X by $\mathcal{Q}^*(X)$ relies on Eq. (17) and is only of interest in the infinite dimensional case where X is unbounded while $\mathcal{Q}^*(X)$ is bounded.

Theorem 11 (Response of fluxes). *Suppose Assumptions 3 hold; that $\sigma(\phi)$ is an integral of parallel transport; the flux \dot{X} and $\mathcal{Q}^*(X)$ are bounded operators. Then, to leading order, the response is memory-less, linear in the driving and given by*

$$\text{Tr}(\dot{X}\rho)(s) = F_\nu(\phi) \dot{\phi}^\nu + O(\varepsilon), \quad \left(\dot{X} = \frac{dX}{dt} \right)$$

(summation implied) and where the response coefficient

$$F_\nu(\phi) = \text{Tr}(\mathcal{Q}^*(X)\partial_\nu\sigma)$$

is a function on \mathcal{M} .

Proof. From Eqs. (15, 51) we have

$$\text{Tr}(\dot{X}\rho) = \varepsilon^{-1} \text{Tr}(\mathcal{L}^*(\mathcal{Q}^*X)\rho) = \text{Tr}(\mathcal{L}^*(\mathcal{Q}^*X)\mathcal{L}^{-1}\dot{\sigma}) = \text{Tr}(\mathcal{Q}^*(X)\partial_\nu\sigma)\dot{\phi}^\nu.$$

□

The observation that one can sometimes avoid computing Green functions in linear response is at the heart of the TKNN formula for the Hall conductance [24].

7.1 Geometric magnetism for iso-spectral families

The response coefficients of an iso-spectral family generated by G_μ are naturally organized as a matrix $F_{\mu\nu}$, relating the response of the flux of G_μ to the driving $\dot{\phi}^\nu$. The analog of the formula in Prop. 10 is

$$\text{Tr}(\dot{G}_\mu \rho)(s) = F_{\mu\nu}(\phi) \dot{\phi}^\nu + O(\varepsilon).$$

Combining Theorem 11 and Eq. (24) we get for the response matrix

$$F_{\mu\nu} = \text{Tr}(\mathcal{Q}^*(G_\mu) \partial_\nu \sigma) = -i \text{Tr}(\mathcal{Q}^*(G_\mu) [G_\nu, \sigma]). \quad (54)$$

We are now ready to state our *main result*:

Theorem 12 (Geometric response). *Suppose G_μ are bounded and $\sigma(\phi) = U(\phi) \sigma U^*(\phi)$ is an integral of parallel transport. Then the response matrix is antisymmetric and given by*

$$F_{\mu\nu} = -i \text{Tr}([G_\mu, G_\nu] \sigma). \quad (55)$$

If, moreover, σ is a projection P then F is the adiabatic curvature of the bundle (see Appendix A):

$$F_{\mu\nu} = i \text{Tr}(P_\perp [\partial_\mu P, \partial_\nu P]). \quad (56)$$

For unitary evolutions, the first part of the theorem reduces to a (special case of) result of Berry and Robbins [10], who coined the term *geometric magnetism* for the anti-symmetric part of F .

The second part of Theorem 12 extends the geometric interpretation of response matrix from the unitary case [7] to open systems. The conditions in the theorem are satisfied for Lindbladians representing relaxation to the ground state and dephasing Lindbladians whose initial state is a spectral projection.

Proof. The conditions have been set so that the formal manipulations are justified

$$F_{\mu\nu} = \text{Tr}(G_\mu \partial_\nu \sigma) = -i \text{Tr}(G_\mu [G_\nu, \sigma]) = -i \text{Tr}([G_\mu, G_\nu] \sigma),$$

where in the second equality we used Eq. (24).

For the second part observe that the equation $-i[G, P] = \dot{P}$ implies

$$P_{\perp}GP = iP_{\perp}\dot{P} \quad \text{and} \quad PGP_{\perp} = -i\dot{P}P_{\perp}.$$

Hence

$$\begin{aligned} F_{\mu\nu} &= -i\text{Tr}([G_{\mu}, G_{\nu}]P) = -i\text{Tr}(PG_{\mu}P_{\perp}G_{\nu}P - PG_{\nu}P_{\perp}G_{\mu}P) \\ &= -i\text{Tr}(\partial_{\mu}PP_{\perp}\partial_{\nu}P - \partial_{\nu}PP_{\perp}\partial_{\mu}P) \\ &= i\text{Tr}(P_{\perp}[\partial_{\mu}P, \partial_{\nu}P]), \end{aligned}$$

where the first line is a readily checked identity. \square

Remark 4. *More details about the dephasing case are in Appendix A. In particular, we give a formula for response when $\sigma(\phi)$ is not the integral of parallel transport.*

In the case that $[G_{\mu}, G_{\nu}]$ is proportional to the identity, the transport coefficients are purely geometric and independent of the dynamics. An example of this kind is the Hall conductance on a torus⁴.

Example 8 (Hall conductance). *Consider the iso-spectral family of Landau Lindbladians, $\{H(\phi), \Gamma_{\pm}(\phi)\}$ of Prop. 8, with $B = 2\pi$ so $[D, D^*] = 4\pi$. By considering the Hilbert space of quasi-periodic functions*

$$\psi(x_1, x_2) = \psi(x_1 + 1, x_2) = e^{2\pi i x_1} \psi(x_1, x_2 + 1)$$

one gets an iso-spectral family of Landau Lindbladians on the unit torus. (The space is invariant under the action of U_{ϕ}). Their stationary state is the thermal state with $4\pi\beta = \log \gamma_{-}/\gamma_{+}$. By Eq. (48) changing the controls (ϕ_1, ϕ_2) may be interpreted as changing the two Aharonov-Bohm fluxes threading the two “holes” of the torus, Fig. 4. The response coefficients have the meaning of conductances. By the first part of Theorem 12

$$F_{\mu\nu} = 2\pi\varepsilon_{\mu\nu} \tag{57}$$

independent of γ_{\pm} . (The insensitivity of the Hall conductance to temperature is a pathology of the model.) In the units we use 2π is the quantum unit of conductance.

An illuminating example of Theorem 12 in the case when $[G_{\mu}, G_{\nu}]$ is not proportional to the identity was recently given by Read and Rezayi [20]. When the G_{μ} are the generators of shears, their commutator is a rotation, hence relating the Hall viscosity to the angular momentum.

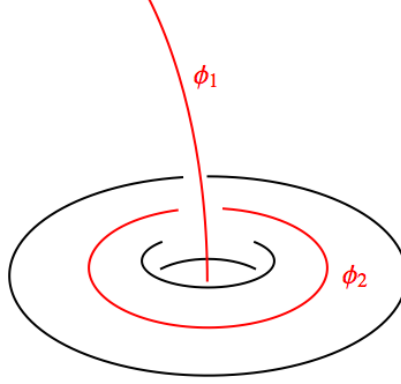


Figure 4: A two dimensional torus threaded by two Aharonov Bohm flux tubes carrying fluxes $\phi_{1,2}$ that serve as controls.

7.2 Friction and dissipation

The fact that $F_{\mu\nu}$ of Eq. (55) is anti-symmetric does not imply the absence of dissipation. It only says that looking at the response of fluxes is not appropriate for the study of dissipation. To explain this statement consider the dissipation associated with the dragging of the anchoring point of a (damped) oscillator coupled to a heat bath at velocity $\dot{\phi}$. The response coefficient relating force to velocity is friction. As there are three forces in the problem—the momentum rate, the force on the anchoring point, and the friction force—there are also three friction coefficients. The friction coefficient associated with the momentum rate vanishes, but the others do not.

Example 9 (Friction: Example 6 continued). *The (unbounded) generator of shifts is, $p = i(a^* - a)/\sqrt{2}$. With σ a thermal state of the oscillator, $\partial_\phi \sigma = -i[p, \sigma]$ is trace class and Eq. (55) applies with $G_\mu = G_\nu = p$. The friction coefficient vanishes, as it must by anti-symmetry.*

The momentum rate vanishes because it can not disentangle the heat lost to the bath from the mechanical work done by the anchoring point. To study dissipation it is not enough to look at the response coefficients of fluxes, nor is it enough to examine the energy of the system.

Indeed, the energy of the (small) system, in the adiabatic limit, is

$$E = \text{Tr}(H\rho) = \text{Tr}(H\sigma) + O(\varepsilon) \quad (58)$$

by Eq. (50). In particular, for an iso-spectral family the energy is constant (to leading order) when H and σ undergo the same unitary transformation, as is the case in the example of the damped oscillator. The energy does not reveal the dissipation.

To reveal the dissipation one needs to look at the breakup of the energy to work and heat. The variation of the energy

$$\delta \text{Tr}(H\rho) = \text{Tr}(H \delta\rho) + \text{Tr}(\delta H \rho) \quad (59)$$

expresses the first law of thermodynamics [22]

$$\delta E = \delta W + \delta Q = (\text{Tr}(\sigma \partial_\mu H) + \text{Tr}(H \partial_\mu \sigma)) \delta\phi + O(\varepsilon).$$

To compute the friction one needs to study the expectation of the spring force $-x$ rather than the momentum flux $\mathcal{L}^*(p)$. (More generally, $\partial_\mu H$ rather than the flux $\mathcal{L}^*(G_\mu)$ of Eq. (34).)

In general, the computation of $\text{Tr}(\rho \partial_\mu H)$ is complicated for two reasons: First, one needs to evaluate \mathcal{L}^{-1} . Second, in the case that the ground state is non-unique, it also needs the explicit expression for the $O(\varepsilon)$ term in the adiabatic expansion, Eq. (50), which are history dependent. For dephasing Lindbladian such a computation is given in [5]. We shall not pursue this direction here.

As a sanity check, let us derive the first law of thermodynamics using the tools of the previous sections. Since H is explicitly time-dependent Assumption 4 does not hold for H , its flux is now made of two terms:

$$\dot{H} = (\partial_\mu H) \dot{\phi}^\mu + \mathcal{L}^*(H). \quad (60)$$

Substituting in Theorems 10, 11 indeed reproduces the first law:

$$\frac{dE}{dt} = (\text{Tr}(\sigma \partial_\mu H) + \text{Tr}(H \partial_\mu \sigma)) \left(\frac{d\phi^\mu}{dt} \right) + O(\varepsilon^2). \quad (61)$$

8 Concluding remarks

We have derived a simple and general formulas for the adiabatic response coefficients for observable of the form $\dot{X} = \mathcal{L}^*(X)$. In the case of iso-spectral families of Lindbladians, the response matrix is determined by geometry and is purely anti-symmetric. We find a range of circumstances where the

response coefficients are given by the adiabatic curvature of the associated stationary projections. It will be interesting to extend the theory to models of extended systems with (non-interacting) fermions.

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A Geometry of projections

Consider continuous orthogonal projections $P_j(\phi)$ with $\sum P_j(\phi) = \mathbb{1}$. The superprojection that takes ρ to $\text{Range } \mathcal{P} = \text{Span}\{P_j\}$ is, Eq. (11),

$$\mathcal{P}(\rho) = \sum_j P_j \rho P_j.$$

We are going to describe parallel transport inside $\text{Range } \mathcal{P}$ [18].

For a given path $P_j(t)$, parallel transport $\dot{u} = \dot{P}_j P_j u$ maps vectors $u(0)$ in the range of $P_j(0)$ to vectors $u(t)$ in that of $P_j(t)$. That map $U(t)$ is unitary and generated by

$$K := iU^* \dot{U} = \sum_j A_j, \quad A_j = i\dot{P}_j P_j. \quad (62)$$

In fact $K^* = K$, since $A_j^* = -iP_j \dot{P}_j = -i\dot{P}_j(1 - P_j) = -i\dot{P}_j + A_j$, and, for U so defined, $u(t) = U(t)u(0)$ satisfies

$$\dot{u} = -iKu = \sum_j \dot{P}_j P_j u,$$

as required. And for $\sigma(t) = U(t)\sigma U^*(t)$ the parallel transport equation (21), $\mathcal{P}(t)\dot{\sigma}(t) = 0$, holds true.

When $\dim P_j = 1$, the parallel transport is manifestly path independent. In general, this is determined by the standard condition of vanishing curvature:

Proposition 13. *Let $\mathcal{A} = (d\mathcal{P})\mathcal{P}$ be an operator valued 1-form. The differential equation*

$$d\sigma = \mathcal{A}\sigma$$

admits a (locally path independent) solution σ if and only if the curvature vanishes

$$\mathcal{R} = 0, \quad \mathcal{R} = -i\mathcal{P} d\mathcal{P} \wedge d\mathcal{P} \mathcal{P}. \quad (63)$$

It implies the following criterion for the case of parallel transport of projections.

Proposition 14 (Adiabatic curvature). *The parallel transport constructed above is locally path independent if and only if the adiabatic curvature*

$$R_{\mu\nu} := \partial_\mu K_\nu - \partial_\nu K_\mu + i[K_\mu, K_\nu] = -i \sum_j P_j [\partial_\mu P_j, \partial_\nu P_j]$$

commutes with all elements in $\text{Range } \mathcal{P}$.

Proof. Parallel transport of a vector u_j in the range of P_j along an infinitesimal square $d\phi^\mu d\phi^\nu$ maps

$$u_j \rightarrow u_j - iR_{\mu\nu}u_j d\phi^\mu d\phi^\nu + o(d\phi^2).$$

The associated adjoint transformation maps the state $\sigma = P_j \sigma P_j$ as

$$\sigma \rightarrow \sigma - i[R_{\mu\nu}, \sigma] d\phi^\mu d\phi^\nu + o(d\phi^2).$$

This allows to read off the curvature $\mathcal{R}_{\mu\nu}$ of \mathcal{P} seen in Eq. (63): By $\mathcal{R} = \mathcal{R}\mathcal{P}$ we have

$$\mathcal{R}_{\mu\nu}\rho = [R_{\mu\nu}, \mathcal{P}\rho].$$

Hence the criterion of vanishing curvature states that R commutes with all elements in $\text{Range } \mathcal{P}$.

Computation gives the commutator

$$[K_\mu, K_\nu] = \sum_j P_{j\perp} [\partial_\mu P_j, \partial_\nu P_j] \quad (64)$$

as the sum of adiabatic curvatures of all the spectral projections. Since

$$\partial_\mu K_\nu - \partial_\nu K_\mu = -i \sum_j [\partial_\mu P_j, \partial_\nu P_j]$$

one finds

$$\partial_\mu K_\nu - \partial_\nu K_\mu + i[K_\mu, K_\nu] = -i \sum_j P_j [\partial_\mu P_j, \partial_\nu P_j].$$

□

For an iso-spectral family of projections

$$P_j(\phi) = \exp(-iG\phi)P_j(0)\exp(iG\phi)$$

the generator of parallel transport, Eq. (62), is

$$K = G - \sum_j P_j G P_j = G - \mathcal{P}^*(G),$$

since $\mathcal{P}^* = \mathcal{P}$ by Eq. (11). While it does not coincide with G it differs from it only inside $\text{Range } \mathcal{P}$

$$\mathcal{Q}^*(G) = \mathcal{Q}^*(K). \quad (65)$$

When $\text{rank } P_j > 1$ and $\dim \mathcal{M} > 1$ the parallel transport can not be integrated in general. And the response coefficients are not functions on the manifold.

Theorem 15. *Suppose G_μ are bounded and \mathcal{L} is a dephasing Lindbladian. Then the response associated to the driving path $\phi(s)$ and flux \dot{G}_μ depends only on the integral of parallel transport $\sigma(\phi)$ and the derivative $\delta\phi$ at the end point,*

$$\text{Tr}(\mathcal{L}^*(G_\mu)\rho(s)) = F_{\mu\nu}\partial_\nu\phi(s),$$

where

$$F_{\mu\nu} = i\text{Tr}([K_\mu, K_\nu]\sigma).$$

Proof.

$$\begin{aligned} F_{\mu\nu} &= \text{Tr}(\mathcal{Q}^*(G_\mu)\partial_\nu\sigma) \\ &= i\text{Tr}(\mathcal{Q}^*(K_\mu)[K_\nu, \sigma]) = i\text{Tr}([K_\mu, K_\nu]\sigma), \end{aligned}$$

where the second line express parallel transport and uses Eq. (65). The last equality is by $\mathcal{P}([K_\nu, \sigma]) = 0$, which characterizes parallel transport and by the way restates Eq. (23). \square

Example 10 (Taming X : Example 5 continued). *Consider a family of Hamiltonians generated by a momentum shift*

$$e^{-i\phi X} H e^{i\phi X} = e^{i\phi T} \otimes a + e^{-i\phi T^*} \otimes a^* = P_+(\phi) - P_-(\phi),$$

where X is the position operator. The generator of the parallel transport is

$$K = i(\dot{P}_+ P_+ + \dot{P}_- P_-) = -\frac{i}{4}[H, \dot{H}] = a^* a - \frac{1}{2}.$$

Although $K \neq X$, their difference commutes with the Hamiltonian. Furthermore K intertwines P_{\pm} ,

$$KP_+ = P_-K,$$

which is equivalent to the statement that K generates no motion inside $\text{Range } P_{\pm}$, $P_{\pm}KP_{\pm} = 0$.

B Currents and unbounded observables

We discuss the precise meaning of Eq. (29) when X is unbounded. We still assume that H and Γ_{α} are bounded, while $X = X^*$ need not be. Yet, the commutators $[H, X]$ and $[\Gamma_{\alpha}, X]$, defined as quadratic forms on the domain $D(X)$ of X , are assumed bounded, and $\Gamma_{\alpha}D(X) \subset D(X)$. Then $\dot{X} = \mathcal{L}^*(X)$ is a bounded operator by natural interpretation of Eq. (25) in the sense of quadratic forms.

Proposition 16. *Under the stated conditions, $\text{Tr}(\dot{X}\sigma) = 0$ for any (trace class) stationary state σ . Moreover, $Q^*(X)$ is well-defined as a bounded operator. It is given as a strong limit, $Q^*(X) = \text{s-}\lim_{n \rightarrow \infty} Q^*(X_n)$, by means of any sequence of bounded approximants X_n with $X_n\varphi \rightarrow X\varphi$, ($\varphi \in D(X)$); finally $\mathcal{L}^*(X) = \mathcal{L}^*(Q^*(X))$.*

Proof. There exist sequences X_n as stated, e.g. $X_n = X/(1 + n^{-1}X^2)$. The assumption states that the bounded operator $[H, X]$ is characterized by the property

$$(\varphi, [H, X]\psi) = (H\varphi, X\psi) - (X\varphi, H\psi), \quad (\varphi, \psi \in D(X)). \quad (66)$$

Hence $[H, X_n] \rightarrow [H, X]$ (weakly), and similarly for $[\Gamma_{\alpha}, X]$. Thus $\mathcal{L}^*(X_n) \rightarrow \mathcal{L}^*(X)$ (weakly). Using that $A_n \rightarrow A$ (weakly) and B trace class imply $\text{Tr}(A_n B) \rightarrow \text{Tr}(AB)$, we conclude

$$\text{Tr}(\mathcal{L}^*(X)\sigma) = \lim_{n \rightarrow \infty} \text{Tr}(\mathcal{L}^*(X_n)\sigma) = 0 \quad (67)$$

by Eq. (27). Moreover, by Eq. (17) we have $\mathcal{L}^*Q^*(X_n) \rightarrow \mathcal{L}^*(X)$. We notice that \mathcal{L}^* is weakly continuous, and so are $(\mathcal{L}^* - z)^{-1}$ and the inverse of \mathcal{L}^* on $\text{Range } \mathcal{L}^* = \text{Range } Q^*$, i.e.

$$(\mathcal{L}^*)^{-1} = -\frac{1}{2\pi i} \oint \frac{dz}{z} (\mathcal{L}^* - z)^{-1} \quad (68)$$

in the notation of Eq. (14). As a result $\mathcal{Q}^*(X_n)$ is weakly convergent to a limit denoted $\mathcal{Q}^*(X)$, and the result follows. \square

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